Nonexistence of Nontrivial Generalized Solutions for 2-D and 3-D BVPs with Nonlinear Mixed Type Equations

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Abstract. A brief survey of known results, open problems and new contributions to the understanding of the nonexistence of nontrivial solutions to nonlinear boundary value problems (BVPs) whose linear part is of mixed elliptic-hyperbolic type is given. Crucial issues discussed include: the role of so-called critical growth of the nonlinear terms in the equation (often related to threshold values of continuous and compact embedding for Sobolev spaces in Lebesgue spaces), the role that hyperbolicity in the principal part plays in over-determining solutions with classical regularity if data is prescribed everywhere on the boundary, the relative lack of regularity that solutions to such problems possess and the subsequent importance to address nonexistence of generalized solutions.

1. INTRODUCTION

Nonexistence results for nonlinear PDEs often involve some kind of critical exponent phenomenon, which is related to criticality in the embedding of the relevant Sobolev space (where one seeks the solution) into some other function space (typically a Lebesgue space).

The Sobolev embedding theorem. Let Ω be a bounded smooth domain in $\mathbf{R}^n, n \in N, n \geq 3$. Then one has the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ with $q \leq \dfrac{2n}{n-2}$.

The critical Sobolev exponent is denoted by $2^*(n) := \dfrac{2n}{n-2}$ and the embedding is compact for $p \in [1, 2^*(n))$, but fails to be compact at the critical exponent.

It is well known, starting from the seminal paper of Pohožaev (1965), that the homogeneous Dirichlet problem for semi-linear elliptic equations such as

$$\Delta u + u|u|^{p-2} = 0 \ \text{ in } \ \Omega,$$

$$u = 0 \ \text{ on } \ \partial\Omega,$$

where Ω is a bounded subset of $\mathbf{R}^n$, with $n \geq 3$, will permit only the trivial solution $u \equiv 0$ if the domain is star-shaped, the solution is sufficiently regular, and $p > 2^*(n) = 2n/(n-2)$.

On the other hand, at subcritical growth in the nonlinearity such as $f(u) = u|u|^{p-2}$, one generically does have existence of solutions. For example, if Ω is a bounded smooth domain in R^n with $n \geq 3$ and if

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 < \dots$$

denote the Dirichlet eigenvalues of the operator $-\Delta$ on $H_0^1(\Omega)$, then variational methods (combined with the maximum principle and regularity theory) yield the following result for the BVP

$$-\Delta u - u|u|^{p-2} = \lambda u \ \text{ in } \ \Omega, \tag{1}$$

$$u = 0 \ \text{ on } \ \partial\Omega. \tag{2}$$

Theorem 1 *In the case $2 < p < 2^*(n)$: for any $\lambda < \lambda_1$ there exists a positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the problem (1) – (2).*

More precise results in the critical case $p = 2^*(n)$ are known, beginning with the celebrated paper of Brezis and Nirenberg [5]. A survey of such results can be found in the monograph of Struwe [39].

Remark 1 *Identities of Pohožaev type have been widely used in the theory PDEs, in particular for establishing non-existence results for large classes of forced elliptic boundary value problems and eigenvalue problems. Let us mention that so-called Pohožaev identities became very popular after the papers of Pucci and Serrin, where a very general form was given in [34] and the relation with the general Noetherian theory is also mentioned. There are hundreds of papers in this approach, connected in very short way with geometrical applications, but we mention only [5], [38]. Or, some recent fractional analogues in [10], [35].*

Remark 2 *Much of the literature in this area concerns elliptic equations. What about the situation for degenerate elliptic equations? While there are many interesting results for particular classes of degenerate elliptic equations, we are far from a general theory. In particular, it would be interesting to have a robust theory that is calibrated to the linear operators studied by G. Fichera and many of his followers as well as the case of operators of non characteristic form, as discussed in the monograph of Oleĭnik, and Radkevič,[27]. See also the more recent work of [36, 37] and [23].*

Equations of mixed elliptic-hyperbolic type. Beginning with the paper [19], critical exponent phenomena have also been studied in this setting. **In the supercritical case**, a **nonexistence principle** was established in [19] for Tricomi type equations in two dimensions with suitable boundary conditions compatible with linear solvability theory. This has led to the study of boundary value problems of the form

$$Lu + F'(u) = K(y)\Delta_x u + \partial_y^2 u + F'(u) = 0 \ \text{ in } \ \Omega, \tag{3}$$

$$u = 0 \ \text{ on } \ \Sigma \subseteq \partial\Omega, \tag{4}$$

where: $\Omega \subset \mathbf{R}^{N+1}$ is a bounded open set with piecewise C^1 boundary; $F'(0) = 0$; L is a mixed type operator of Gellerstedt type with $K(y) = y|y|^{m-1}, m > 0$ is a pure power type change function; $x \in \mathbf{R}^N$ with $N \geq 1$.

All such operators are invariant with respect to a certain anisotropic dilation which defines a suitable notion of star-shapedness by using the flow of the vector field which is the infinitesimal generator of the invariance. The complete symmetry group for such operators has been studied in [20] and connected to a singular geometric structure of mixed Riemannian-Lorentzian signature in [26].

In dimension 2 (where $N = 1$), such operators have a long standing connection with transonic fluid flow, a connection first established by Frankl' (1945) [11]. Both classical and modern surveys of such applications can be found in papers of Cathleen Morawetz [24, 25].

Two different classes of boundary value problems are to be considered in the mixed type setting. They differ as to whether $\Sigma = \partial\Omega$ or is a proper subset of $\partial\Omega$ and are called **closed** and **open** boundary conditions, respectively.

1. The nonexistence principle for the (closed) Dirichlet problem:

We need an additional geometric hypothesis on the boundary – the hyperbolic portion of the boundary is **sub-characteristic** for the operator L. This condition is natural for applications such as transonic flow (see [14, 39]).

2. For the open boundary problems, the situation is more difficult.

The lack of a boundary condition on a part $\Gamma \subsetneq \partial\Omega$ complicates the control of the corresponding boundary integral in the Pohožaev argument, but if Γ is characteristic and tangential to the dilation flow, a sharp Hardy-Sobolev inequality with reminder term successfully applied in the critical case, or for the generalized solutions (see [8, 9, 22]) ensures that the contribution along Γ has the right sign.

In all cases, for the operator L the critical exponent phenomenon is of pure power type of order p where p agrees with a critical Sobolev exponent in the embedding of a suitably weighted version of $H^1_0(\Omega)$ into $L^p(\Omega)$.

The point of this note is:

1. To recall some known results, published or merely announced.
2. To announce that the nonexistence principle is valid for a large class of such problems, even in higher dimensions and in some critical cases.
3. To stress that such nonexistence principles can hold also for generalized solutions. This is important since boundary value problems for even linear mixed type equations usually have a limited degree of smoothness.

2. THE DIRICHLET PROBLEM FOR MIXED TYPE NONLINEAR EQUATIONS

In this section, we recall the relatively simple result that the closed Dirichlet problem for the supercritical semi-linear Gellerstedt equation admits only trivial classical solutions. Consider the problem:

$$Lu + F'(u) = y|y|^{m-1}\Delta_x u + \partial_y^2 u + F'(u) = 0 \text{ in } \Omega, \tag{5}$$

$$u = 0 \text{ on } \partial\Omega, \tag{6}$$

where $\Omega \subset \mathbf{R}^{N+1}$ is a **bounded mixed domain**; that is, an bounded, open and connected subset of with piecewise C^1 boundary such that

$$\Omega \cap \mathbf{R}^{N+1}_{\pm} \neq \emptyset$$

where $\mathbf{R}^{N+1}_{\pm} = \{(x, y) \in \mathbf{R}^N \times \mathbf{R} : \pm y > 0\}$ are the **elliptic/hyperbolic half-spaces for** L. One calls $\Omega_\pm = \Omega \cap \mathbf{R}^{N+1}_{\pm}$ the **elliptic/hyperbolic regions** in Ω.

The hyperbolic boundary $\Sigma_- = \partial\Omega \cap \mathbf{R}^{N+1}_{-}$ will be called **sub-characteristic** for the operator L if one has

$$y|y|^{m-1}|\nu_x|^2 + \nu_y^2 \geq 0, \text{ on } \Sigma_-, \tag{7}$$

where $\nu = (\nu_x, \nu_y)$ is the (external) normal field on the boundary. If (7) holds in the strict sense, we will call Σ_- **strictly sub-characteristic**, which just means that Σ_- is a piece of a **spacelike hypersurface** for the operator L.

The operator L is invariant with respect to the anisotropic dilation whose infinitesimal generator is

$$V = -\sum_{j=1}^{N}(m+2)x_j\partial_{x_j} - 2y\partial_y \tag{8}$$

The dilation is used to define a class of admissible domains for the nonexistence principle in the following way.

Definition 1 *One says that Ω is **V-star-shaped** if for every $(x_0, y_0) \in \overline{\Omega}$ the time t flow of (x_0, y_0) along V lies in $\overline{\Omega}$ for each $t \in [0, +\infty]$.*

A simple application of the divergence theorem shows that ff Ω is V-star-shaped then $\partial\Omega$ will be **V-star-like** in the sense that on $\partial\Omega$ one has

$$((m+2)x, 2y) \cdot \nu \geq 0. \tag{9}$$

If the inequality (9) holds in the strict sense, we will say that $\partial\Omega$ is **strictly V-star-like**.

The dilation generated by V in (8)) also gives rise to a critical exponent

$$2^*(N, m) = \frac{2[N(m+2)+2]}{N(m+2)-2}$$

for the embedding of the weighted Sobolev space $H^1_0(\Omega; m)$ into $L^p(\Omega)$ where

$$\|u\|^2_{H^1_0(\Omega;m)} := \int_\Omega \left(|y|^m|\nabla_x u|^2 + u_y^2\right) dxdy$$

defines a natural norm for which to begin the search for weak solutions. The basic result we wish to recall is the following.

Theorem 2 *([21]) Let Ω be mixed type domain which is star-shaped with respect to the generator V of the dilation invariance and whose hyperbolic boundary is sub-characteristic. Let $u \in C^2(\overline{\Omega})$ be a solution to the Dirichlet problem with $F'(u) = u|u|^{p-2}$. If $p > 2^*(N,m)$, then $u \equiv 0$. In addition, if $\partial\Omega$ is strictly V-star-like, then $u \equiv 0$ also holds for $p = 2^*(N,m)$.*

The result follows from the application of a Pohožaev type identity calibrated to the dilation invariance combined with an integral identity obtained by multiplying the PDE by u_y.

On the other hand, for asymptotically linear f, one has many recent existence results for nonlinear Dirichlet problems involving mixed type operators as one can find in [16, 17].

3. HARDY SOBOLEV INEQUALITIES

In contrast with the (closed) Dirichlet problems, in boundary value problems with open boundary conditions for mixed type equations, the absence of a boundary condition on some part of the boundary leaves additional boundary integrals in Pohožaev type identities whose sign must be controlled. A key role is played by **weighted versions of the Hardy inequality** ([19, 20]). A prototype of such an inequality (often referred to as a Hardy-Sobolev inequality) is:

$$\frac{(\alpha-1)^2}{4}\int_0^R t^{\alpha-2}w^2(t)\,dt \;\le\; \int_0^R t^{\alpha}[w'(t)]^2\,dt\,, \tag{10}$$

which holds for every $w \in C^1(0,R) \cap C([0,R])$ such thats $w(R) = 0$ and $\alpha > 1$. The constant is sharp. Moreover, the value of the sharp constant is needed and used in the proof of even the first nonexistence results in [19]. One should note that there are many multidimensional analogues of (10).

Further progress in the cases of critical growth rely on refinements of (10) such as the **Hardy-Sobolev inequality with remainder term**; refinements which have their origins in the work of Brezis-Vasquez [6] and Chen-Shen [7]. Such inequalities have been successfully applied for critical growth mixed type equations (even with generalized solutions) in [8, 9, 22]. A prototype of such a refined inequality is the following result: Let $w \in C^1(0,R) \cap C([0,R])$ satisfy $w(R) = 0$ and $\alpha > 1$. Then

$$\frac{4}{R^2}\int_0^R t^{\alpha}w^2(t)\,dt + \frac{(\alpha-1)^2}{4}\int_0^R t^{\alpha-2}w^2(t)\,dt \le \int_0^R t^{\alpha}[w'(t)]^2\,dt.$$

4. TWO DIMENSIONAL BOUNDARY VALUE PROBLEMS

In this section, we briefly explain the main results in [19] and [21] on the nonexistence principle for mixed type equations in two dimensional problem, where the theory is most complete. We consider the two-dimensional version of the main problem (5) – (6)

$$Lu + F'(u) \equiv K(y)u_{xx} + u_{yy} + F'(u) = 0 \;\text{ in }\; \Omega, \quad K(y) = y|y|^{m-1}, \tag{11}$$

$$u = 0 \;\text{ on }\; \Sigma, \tag{12}$$

where $m > 0$, Ω is a mixed-type domain in the plane, and $\Sigma \subset \partial\Omega$.

As noted in section 2, a portion Σ_1 of the hyperbolic boundary $\Sigma_- = \partial\Omega \cap \{y < 0\}$ will be called *sub-characteristic* for the operator L if one has

$$K(y)\nu_x^2 + \nu_y^2 \ge 0 \;\text{ on }\; \Sigma_1, \tag{13}$$

where $\nu = (\nu_x, \nu_y)$ is again the external normal field at the boundary and Σ_1 *strictly sub-characteristic* if (13). holds in the strict sense.

As noted, the operator L in (11) is invariant with respect to the anisotropic dilation whose infinitesimal generator in dimension two is

$$V = -(m+2)x\partial_x - 2y\partial_y. \tag{14}$$

The notions of Ω being V-*star-shaped* and $\partial\Omega$ being V-*star-like* are given in Definition 1 and formula (9) of section 2.

The dilation generated by V in (14) also gives to a critical exponent

$$2^*(1,m) = \frac{2(m+4)}{m} \tag{15}$$

for the embedding of the weighted Sobolev space $H_0^1(\Omega;m)$ into $L^p(\Omega)$, where

$$\|u\|^2_{H_0^1(\Omega;m)} := \int_\Omega \left(|y|^m u_x^2 + u_y^2\right)\, dxdy \tag{16}$$

defines a natural norm for variational solutions. Notice that (11) is the Euler-Lagrange equation associated with the functional $J(u) = \int_\Omega (\mathcal{L}(y,\nabla u) + F(u))\, dxdy$, where

$$\mathcal{L}(y,\nabla u) = \frac{1}{2}(K(y)|\nabla_x u|^2 + u_y^2) \tag{17}$$

is the Lagrangian corresponding to L.

In [19], one has the nonexistence principle for the *Tricomi problem* in the domain Ω which consists of an elliptic part $\Omega \cap \{y > 0\}$ and the characteristic "triangle" as the hyperbolic part $\Omega \cap \{y < 0\}$. In this case $\partial\Omega = \Sigma \cup \Gamma$ with $\Sigma = \sigma \cup AC$ and $\Gamma = BC$, where σ is an arc in the elliptic region $\Omega \cap \{y > 0\}$ and AC/BC are characteristics of L in $\Omega \cap \{y < 0\}$ with negative/positive slopes, respectively, intersecting in $C = (x_C, y_C)$; that is, with $x_A < x_B$

$$AC := \left\{(x,y) \in \mathbf{R}^2 : \ \ y_C \le y \le 0;\ (m+2)(x - x_A) = 2(-y)^{(m+2)/2}\right\}, \tag{18}$$

$$BC := \left\{(x,y) \in \mathbf{R}^2 : \ \ y_C \le y \le 0;\ (m+2)(x - x_B) = -2(-y)^{(m+2)/2}\right\}. \tag{19}$$

In [21], two other problems (with uniqueness results for the linear equation) are considered. The first is the *Frankl' problem* in which one characteristic arc, say AC, in the Tricomi case is replaced by a sub-characteristic arc Γ_1 connecting A to some point (still called C), on the characteristic BC of positive slope through B. We call such a domain a *Frankl' domain*. Actually, the *Frankl' domain* is a generalization of a *Tricomi domain* in the case when $\Gamma_1 = AC$ is everywhere characteristic. Since the operator $L = y|y|^{m-1}\partial_x^2 + \partial_y^2$ is invariant with respect to translations in x, we may assume, without loss of generality, that the point $B = (x_B, 0) \in \overline{\Omega}$ with maximal x-coordinate on the parabolic segment is situated at the origin, that is, $B = (0,0)$.

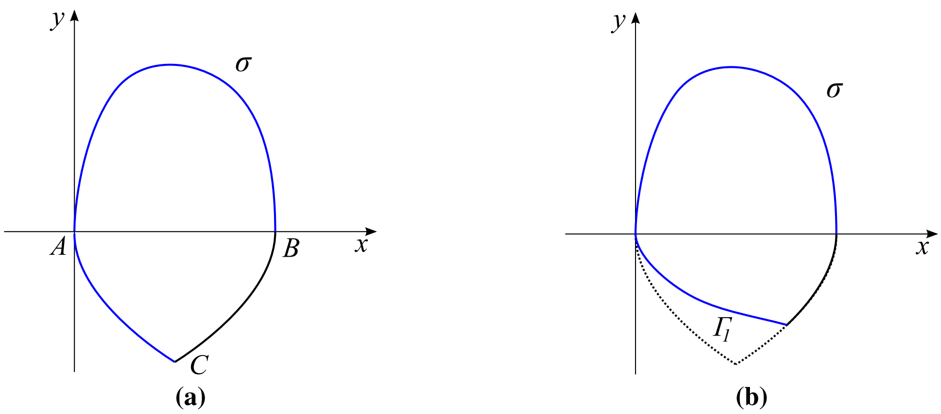

FIGURE 1. **(a)** Tricomi Problem. **(b)** Frankl' Problem.

Theorem 3 *([19], [9] in the critical case)* Let $\Omega \subset \mathbf{R}^2$ be a Frankl' domain with boundary $\partial\Omega = \sigma \cup \Gamma_1 \cup BC$ (with Γ_1 sub-characteristic for L). Assume that Ω is star-shaped with respect to the generator V as in (14) of the dilation invariance for L. Let $u \in C^2(\overline{\Omega})$ be a solution to (11) – (12) with $F'(u) = u|u|^{p-2}$. If $p > 2^*(1,m) = 2(m+4)/m$ (the critical Sobolev exponent), then $u \equiv 0$. Suppose in addition that $\sigma \cup \Gamma_1$ is strongly star-like at its noncharacteristic points. Then if $p = 2(m+4)/m$, then $u \equiv 0$.

The same arguments work also for the *Guderley-Morawetz problem* in which one removes a *solid backward light cone with vertex at the origin*

$$\overline{\mathcal{K}}(0) = \{(x, y) \in \mathbf{R}^2 : \ (m+2)^2 x^2 \leq 4(-y)^{m+2}, \ y \leq 0\} \tag{20}$$

from a bounded open and simply connected set $\widetilde{\Omega}$ which contains the origin. Let $\Omega = \widetilde{\Omega} \setminus \mathcal{K}(0)$ be the resulting domain, called a *Guderley-Morawetz domain*. Its boundary consists of $\sigma \cup \Gamma_1 \cup \Gamma_2 \cup BC_1 \cup BC_2$ where σ is the elliptic boundary which connects A_1 with A_2 on the parabolic line, Γ_j are sub-characteristic arcs descending from A_j which intersect the characteristics BC_j forming the boundary of (2.10) at the points C_j. The boundary value problem is to solve (11) – (12), where in this case $\Sigma = \sigma \cup \Gamma_1 \cup \Gamma_2$.

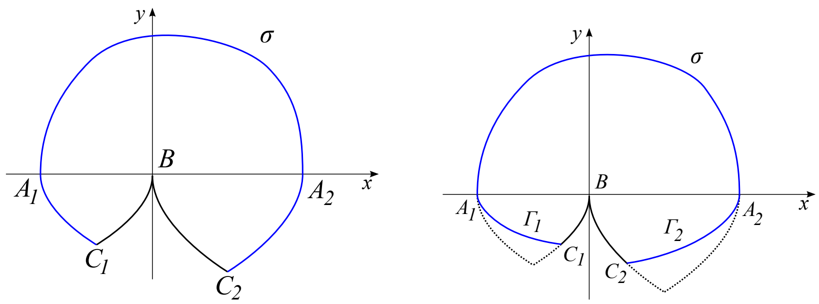

FIGURE 2. Guderley-Morawetz Problems.

Now we summarize the previous results ([21]) in the supercritical case, complementing them with results from [8], and, especially, incorporating also the critical case $p = 2^*(1, m)$.

Theorem 4 Let $\Omega \subset \mathbf{R}^2$ be a Guderley-Morawetz domain with boundary $\sigma \cup \Gamma_1 \cup \Gamma_2 \cup BC_1 \cup BC_2$ (where Γ_1, Γ_2 are sub-characteristic). Assume that $\sigma \cup \Gamma_1 \cup \Gamma_2$ is star-like with respect to the generator V *(see* 14) of the dilation invariance for L. Let u$\in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution to (11) – (12) with $F'(u) = u|u|^{p-2}$. If $p > 2^*(1, m) = 2(m+4)/m$ (the critical Sobolev exponent), then $u \equiv 0$. *Let* we suppose in addition that $\sigma \cup \Gamma_1 \cup \Gamma_2$ is strongly star-like at its noncharacteristic points. Then if $p = 2^*(1, m) = 2(m+4)/m$, then $u \equiv 0$.

Remark 3 *In the linear case, the boundary-value problems of Tricomi, Frankl', and Guderley-Morawetz are the classical boundary value problems that appear in hodograph plane for 2-D transonic potential flows ([4], [25]). The first two of these problems are relevant to flows in nozzles and jets, and the third problem occurs as an approximation to a respective "exact" boundary-value problem in the study of flows around airfoils (in this case the 'exact' problem is the "closed" Dirichlet problem). Existence of* weak solutions *and uniqueness of* strong solutions *in weighted Sobolev spaces for the Guderley-Morawetz problem were first established by Morawetz [24] by reducing the problem to a first order system which then gives rise to solutions to the scalar equation in the presence of sufficient regularity. The availability of such sufficient regularity follows from the work of Lax and Phillips [15] who also established that the weak solutions of Morawetz are strong (see also Remark 5 below). For all of these linear problems there are suitable uniqueness theorems which, as we have shown, imply a Pohožaev nonexistence principle for supercritical nonlinear variants, provided that the domains are suitably star-shaped. The key component of the proof of this implication is that the boundary curves, on which there is no imposed boundary condition, are characteristics which are tangential to the flow generated by the dilation invariance.*

Remark 4 *With respect to existence, one knows that weak solutions exist for the linear problem under suitable hypotheses on* Σ. *In particular, one has weak existence and uniqueness in* $H^1(\Omega)$ *(i.e.* $H^1(\Omega; m)$ *with norm (16)) for* angular (normal) *domains in which the elliptic arc meets the parabolic line at acute (right) angles. Such type of results for "closed" boundary-value problems with Dirichlet or mixed Dirichlet-conormal boundary data can be found in recently published [18].*

Remark 5 *With respect to the regularity of solutions, we have assumed throughout this work that the solutions are of class* $C^2(\Omega) \cap C^1(\overline{\Omega})$, *which is clearly too much. In fact, even for problems with open boundary conditions, it is generally expected to have the possibility of isolated singularities in the first derivatives at parabolic boundary* *points. On the other hand, it would be of considerable interest to establish, if possible, the nonexistence of non-trivial solutions for supercritical problems in the class of* strong solutions $u \in H^1_\Sigma(\Omega; m)$ *in the sense that: there exists a sequence* $\{u_j\} \subset C^2(\overline{\Omega})$, $u_j = 0$ *on* Σ, *such that*

$$\lim_{j\to+\infty} \|u_j - u\|_{H^1(\Omega;m)} = 0, \lim_{j\to+\infty} \|Lu_j - F'(u_j)\|_{L_2(\Omega)} = 0.$$

Essentially, this is a problem in linear analysis, of the type "weak solutions = strong solutions" as considered by Lax and Phillips [15] and others.

5. THE GOURSAT 2-D PROBLEM FOR NONLINEAR TRICOMI EQUATION IN THE FRAMEWORK OF GENERALIZED SOLUTIONS

In our discussion up to now, strong regularity $u \in C^2(\overline{\Omega})$ was used in the proof of nonexistence results based on Pohožaev type identities calibrated to invariances in the linear part of the equation. This is an unpleasant state of affairs, since even for linear equations one does not expect such regularity in the solutions and one might worry that the nonexistence comes from such a high regularity demand. However, the accepted thinking is that this is a technical matter and that critical and supercritical growth ought to impose triviality for a suitable notion of weak solutions in V-star-shaped domains. This is indeed the case for simple, but important planar problems involving degenerate hyperbolic equations, as was shown in [22]. We wish to briefly describe the idea here.

One considers the problem

$$\begin{cases} Lu + F'(u) = 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \Gamma = AC \cup AB, \end{cases} \tag{21}$$

where $L = -T = yD_x^2 + D_y^2$ is the Tricomi operator, $A = (-2x_0, 0), B = (0,0), C = (-x_0, -(3x_0/2)^{2/3})$ and $f = F' \in C^0(\mathbf{R})$ with primitive $F(s) := \int_0^s f(t)\,dt$ satisfying

$$F(0) = 0, \tag{22}$$

where the translation invariance in the equation means that $B = (0,0)$ is a harmless normalization. A solid linear theory for the equation $Lu = f$ with $f \in L^2(\Omega)$ is developed in [22], including well-posedness in the natural Sobolev space $H^1_\Gamma(\Omega)$ which is the completion in the norm

$$\|\psi\|_{H^1(\Omega)} = \left(\int_\Omega (\psi_x^2 + \psi_y^2 + \psi^2)\,dxdy \right)^{1/2}$$

of the space

$$C^\infty_\Gamma(\overline{\Omega}) = \left\{ \psi \in C^\infty(\overline{\Omega}) : \psi \equiv 0 \text{ on } N_\epsilon(\Gamma) \ \text{ for some } \epsilon > 0 \right\},$$

where $N_\epsilon(\Gamma)$ is an ϵ neighborhood of $\Gamma = AC \cup AB$. Elements of the linear theory include well-posedness, elements of a spectral theory, partial regularity results and maximum and comparison principles. For the nonlinear problem (21), existence of weak solutions in $H^1_\Gamma(\Omega)$ with nonlinearities of unlimited polynomial growth at infinity is proven by combining standard topological methods of nonlinear analysis with the linear theory.

On the other hand, for homogeneous supercritical nonlinearities (when $F'(0) = 0$), the uniqueness of the trivial solution in the class of weak solutions $H^1_\Gamma(\Omega)$ is obtained. More precisely, consider $F' = f$ of pure power form

$$f(s) = s|s|^{p-2} \tag{23}$$

with supercritical growth

$$p \geq 2^*(1,1) = 10. \tag{24}$$

One has the following result.

Theorem 5 *([22]) If $f = F'$ satisfies (23)–(24), then the only solution $u \in H^1_\Gamma(\Omega)$ of (21) is the trivial solution* $u = 0$.

For $u \in C^2(\overline{\Omega})$, this was known from [19] when $p > 2^*(1,1) = 10$ and from [21] when $p = 2^*(1,1) = 10$. The proof for generalized solutions $u \in H^1_\Gamma(\Omega)$ combines suitable Pohoẓaev type identities with well tailored mollifying procedures, which exploit the simple form of the domain which is a characteristic triangle ABC.

6. THE PROTTER-MORAWETZ MIXED TYPE PROBLEM IN HIGHER DIMENSIONS

In this section, we study a generalization of the semi-linear Guderley-Morawetz planar problem to higher dimensions. More precisely, we consider the problem

$$Lu + F'(u) = 0 \ \text{ in } \ \Omega, \tag{25}$$

$$u = 0 \ \text{ on } \ \Sigma, \tag{26}$$

where L is the Gellerstedt operator $L = K(y)\Delta_x + \partial_y^2$ on a bounded open mixed-type domain $\Omega \subset \mathbf{R}^{N+1}$, $N \geq 2$, and the hyperbolic part of the domain $\Omega_- = \Omega \cap \{y < 0\}$ has the particular form

$$\Omega_- = \left\{(x, y) \in \mathbf{R}^N \times \mathbf{R} : \ \frac{2}{m+2}(-y)^{(m+2)/2} \leq |x| \leq R - \frac{2}{m+2}(-y)^{(m+2)/2}, \ y \leq 0\right\} \tag{27}$$

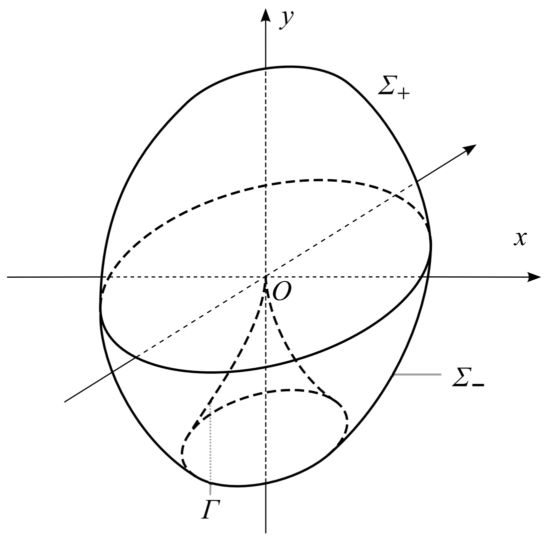

FIGURE 3. Protter-Morawetz Problem.

The "lateral boundaries" of Ω_- are characteristic surfaces (see Figure 3). The outer part Σ_1, where $(m+2)^2(|x| - R)^2 = 4(-y)^{m+2}$, is the boundary of the domain of dependence of the point $(0, -((m+2)R/2)^{2/(m+2)})$, while the inner part Γ, where $(m+2)^2|x|^2 = 4(-y)^{m+2}$, is the boundary of the backward light cone with vertex at the origin. Such a domain will be called a *Protter domain,* and the *Morawetz-Protter problem* consists in prescribing boundary data on the entire elliptic boundary Σ_+ and the portion Σ_- of the hyperbolic boundary. Murray Protter (1954) proposed these boundary conditions in three dimensions (when $N = 2$) for the linear equation (cf. [33]) as an analog to the planar Guderley-Morawetz problem, but even in the linear case a complete general understanding of the situation is not clear even now. Here it is shown that for the semi-linear Morawetz-Protter problem with the Gellerstedt equation the nonexistence principle is valid in any dimension.

Theorem 6 *Let $\Omega \subset \mathbf{R}^{N+1}$ be a Protter domain with boundary $\Sigma_+ \cup \Sigma_1 \cup \Gamma$. Assume that Ω is star-shaped with respect to the generator V of the dilation invariance for L. Let $u \in C^2(\overline{\Omega})$ be a solution to (25) – (26) with $\Sigma = \Sigma_+ \cup \Sigma_-$ and $F'(u) = u|u|^{p-2}$. If $p > 2^*(N, m)$ (the critical Sobolev exponent, then $u \equiv 0$. If, in addition, Σ_+ is strictly V-star-like, then the result holds also for $p = 2^*(N, m)$.*

Remark 6 *Let us mention that this result was proved in [21] for the supercritical case. In the critical case the result was announced in [9].*

Outline of the idea of the proof: The results follow from integral identities of Pohožaev type which are suitably calibrated to an invariance with respect to anisotropic dilations in the linear part of the equation. At critical growth, the nonexistence principle is established by combining the dilation identity with another energy identity. Now, the set Γ on which no data is placed is a piece of a characteristic surface. The lack of a boundary condition on Γ complicates the control of the corresponding boundary integral in the Pohožaev argument, but if Γ is characteristic and tangential to the dilation flow, a sharp Hardy-Sobolev inequality is used to control the terms in the integral identity corresponding to the lack of a boundary condition, as was first done in [19] and later in [21].

7. THE PROTTER WEAKLY HYPERBOLIC PROBLEM

Following Protter [33], we will consider a generalization of the semi-linear $2D$ Goursat problem to higher dimensions. More precisely, we consider the problem

$$Lu + u|u|^{p-2} = 0 \text{ in } \Omega_-, \tag{28}$$

$$u = 0 \quad \text{on} \quad \Sigma_1, \; u_y = 0 \quad \text{on} \quad \Sigma_0 := \{y = 0\} \cap \bar{\Omega}_-, \tag{29}$$

where $L = K(y)\Delta_x + \partial_y^2$ is the Gellerstedt operator (or the wave operator for $m = 0$) on a hyperbolic domain Ω_-, given by (27). Here we show that the nonexistence principle is valid for the semi-linear Protter problem for the Gellerstedt equation (or wave equation for $m = 0$) in any dimension.

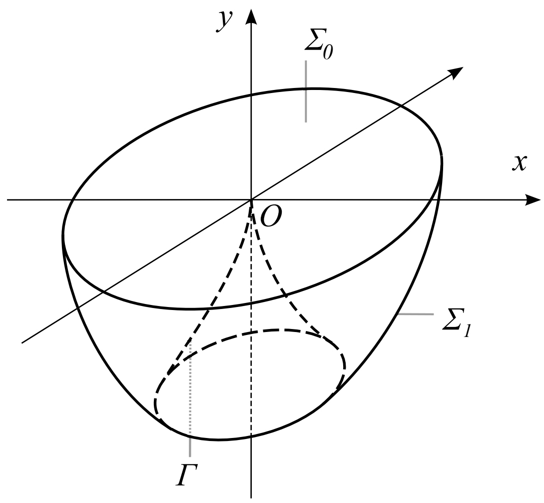

FIGURE 4. Protter hyperbolic Problem.

Actually, $\Sigma_0 := \{y = 0\} \cap \{|x| \leq R\}$ and mention that the "lateral boundaries" of Ω are characteristics (or more generally subcharacteristic) surfaces.

The dilation gives rise to a critical exponent

$$2^*(N, m) = \frac{2[N(m+2)+2]}{N(m+2)-2} \tag{30}$$

for the embedding into $L^p(\Omega)$ of the weighted Sobolev space $\tilde{H}^1(\Omega; m)$ of functions from $H^1(\Omega; m)$, that satisfy the boundary conditions (29)

$$u = 0 \quad \text{on} \quad \Sigma_1, \; u_y = 0 \quad \text{on} \quad \Sigma_0$$

equipped by the norm

$$\|u\|^2_{\tilde{H}^1(\Omega;m)} := \int_\Omega \left(|y|^m |\nabla_x u|^2 + u_y^2\right) dxdy$$

It also defines a natural norm for the search for weak solutions in a variational formulation of the problem.

Remark 7 *It is important to note that if we are looking for the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$, the exponent $2^*(N,m)$ is the exact one. Actually, we have*

$$2^*(N,m_1) < 2^*(N,m) < 2^*(N,0),$$

if $m_1 > m > 0$. If we are looking for embedding of some wider functional space (for example, $\tilde{H}^1(\Omega;m)$), the embedding into $L^q(\Omega)$ is not obviously fulfilled and has to be studied additionally. In the case of BVP (28) – (29) the following two results are not to hard to prove:

1. *The embedding of $\tilde{H}^1(\Omega;m)$ into $L_q(\Omega)$ for $q \leq 2^*(N,m)$ is true.*
2. *For any $q > 2^*(N,m)$ there exists a function $u_q \in \tilde{H}^1(\Omega;m)$, which does not belong to $L_q(\Omega)$.*

Lemma 7 *The embedding number given by (30) is the exact critical number of embeddng of $H^1_\Sigma(\Omega;m)$ to $L^p(\Omega_-)$.*

In [9] we annonced the following result.

Theorem 8 *Let $\Omega \subset \mathbf{R}_-^{N+1}$ be a Protter domain with boundary $\Sigma_0 \cup \Sigma_1 \cup \Gamma$. Let $u \in C^2(\overline{\Omega_-})$ be a solution to the Protter weakly hyperbolic problem for $m \geq 0$ and $F'(u) = u|u|^{p-2}$.*

If $p \geq 2^*(N,m)$, then $u \equiv 0$.

Remark 8 *Even in the linear case, the question of well-posedness is surprisingly subtle and not completely resolved (see [32] , [1] and [2] and the references cited therein). One has uniqueness results for* quasi-regular *solutions [3], a class of solutions introduced by Protter, but there are real obstructions to existence in this class. To explain the situation for the Morawetz-Protter elliptic-hyperbolic Problem from Section 2, let us begin with a simpler question, as follows. What do we know about the Protter problem in Ω_- (which is the natural analogue to the two dimensional degenerate hyperbolic Darboux problem, i.e., the domain Ω_- consists only of the hyperbolic part of Ω. In [32] it was shown that the homogeneous adjoint problem admits infinitely many nontrivial classical solutions $v_n \in C^n(\overline{\Omega_-})$, $n \in \mathbf{N}$. This implies that for classical solvability of the linear Protter problem in a hyperbolic domain Ω_- there are an infinite number of side conditions of the form $f \perp v_n$ which must be satisfied by the right-hand side f in the equation. The concept of a* generalized solution *with a possible singularity on the inner cone Γ was introduced in [32] and a relevant result is the weak well-posedness in this class when $\Omega = \Omega_-$. In addition, [32] contains the construction of a sequence of unique generalized solutions u_n with $Lu_n \in C^n(\overline{\Omega})$ but which exhibit a strong singularity at the vertex of the cone Γ. The order of the strong singularity of u_n grows with n to infinity. For the case of mixed-type elliptic-hyperbolic domain Ω the problem of existence of weak solutions for a large class of f remains still open. It is also an open question whether the strong ill-posedness described above for a hyperbolic domain Ω_- is also present in mixed-type elliptic-hyperbolic domains Ω? Finally, we note that uniqueness for the Protter problem in $\Omega_- \subset \mathbf{R}^4$ in the case of the wave equation ($m = 0$) has been shown by Garabedian [12]. Some recent results for the same case of the wave equation, concerning the exact asymptotic behavior of singularities of the generalized solution of the Protter problem above ($m = 0$), are found in [31, 30].*

Remark 9 *With respect to the regularity of solutions, we have assumed throughout the present work (excluding section 5) that the solutions are of class $C^2(\Omega) \cap C^1(\overline{\Omega})$, which is clearly too much. In fact, even for problems with open boundary conditions one expects, in general, to have the possibility of isolated singularities in the first derivatives at parabolic boundary points. This is compatible with weighted Sobolev spaces, like $\tilde{H}^1(\Omega;m)$, in which it is natural to find solutions. Moreover, for semilinear problems of the form (28) – (29) with subcritical growth and with Gellerstedt operator L, known existence results are usually – but not always – for* generalized solutions, *in the sense that, there exists $u \in H^1_\Sigma(\Omega;m)$ for which*

$$\int_\Omega \left[K(y)(\nabla_x u \cdot \nabla_x \varphi) + u_y \varphi_y + F'(u)\varphi\right] dxdy = 0 \quad \forall \varphi \in H^1_{\Sigma_1 \cup \Gamma}(\Omega;m) \tag{31}$$

where $\partial\Omega = \Sigma \cup \Gamma = \Sigma_1 \cup \Sigma_{\text{char}} \cup \Gamma$ with Γ being a piece of a characteristic surface not carrying the boundary condition, and Σ_1 being the noncharacteristic part of the boundary carrying the boundary condition. The respective Sobolev spaces are the closures of smooth functions vanishing near the relevant boundary portion with respect to the norm.

Remark 10 *Actually, if in the definition of the generalized solution (see (31) we use only test functions which are zero in a neighborhood of the point O, it allows to find singular solutions with very strong singularity at the point O. This singularity is isolated at the point O only. In the linear case this have been done in many papers (see for example [31, 30]. Currently, we have constructed a singular solution for the same linear Protter problem for wave equation (m=0) with exponential growth of singularity at the point O(see [13]). Also, some analogous results for some singular solutions in the linear case of Keldysh type equation were obtained, see: [28], [29].*

8. GENERALIZED SOLUTIONS OF WEAKLY HYPERBOLIC PROTTER PROBLEM

Here we are looking for different kinds of solutions of the homogeneous Protter weakly hyperbolic problem (28) – (29).

Definition 2 *The function $u(x, y)$ is a **generalized solution** of problem (28) – (29) iff $u \in H^1(\Omega) \cap L^p(\Omega)$, $u = 0$ on Σ_1 and*

$$\int_\Omega \left[u_y v_y - (-y)^m \sum_{j=1}^{N} u_{x_j} v_{x_j} - u|u|^{p-2} v \right] dxdy = 0,$$

holds for each function $v \in C^1(\overline{\Omega})$, $v = 0$ on Γ, $v_y = 0$ on Σ_0 and in a neighborhood of the origin $O(0, \ldots, 0)$.

Theorem 9 *Let $\Omega \subset \mathbf{R}^{N+1}$ be a Protter domain with boundary $\Sigma_0 \cup \Sigma_1 \cup \Gamma$. Let $u(x, y)$ be a generalized solution to (28) – (29) for $m \geq 0$ and $F'(u) = u|u|^{p-2}$. Let, in addition, $u \in L^{2p}(\Omega)$.*
If $p \geq 2^*(N, m)$, then $u \equiv 0$.

SCETCH OF THE PROOF:

The idea is to use the Pohožaev identities again. For this reason, we will approximate a generalized solution by smooth functions and the difficulties are two:

1. Because of the lack of smoothness of the generalized solution *u(x,t)*, we will need a suitable mollification procedure. However, because of the shape of the domain Ω_-, there appear some technical difficulties in using the techniques we have developed thus far around the singular point O. One promising idea is to begin the proof by considering a sequence of cutoff functions u_k(x,t), which vanish on small sets around O. Using a special mollifier technique $(u_k)_\epsilon$(x,t) along the sequence of these cutoff functions, and controlling the errors as $k \to +\infty$ seems a feasible task and we are confident that this will lead to the needed Pohožaev identities along the approximating sequence of smooth functions.

2. Finally, using Caratheódory theory with a special partition of unity, we find a smooth approximation of the "cutoff general solution " u_k(x,t).

3. The last step is to approve that both approximations (cutoff and mollifier) converge and, as a result, the generalized solution will be zero in Ω_-.

4. Let us mention here that in the critical case, due to the lack of smoothness of the general solution u(x,t), the weighted Hardy inequality with remainder term needs to be involved.

9. OPEN PROBLEMS

1. What happens with other Problems, formulated above, in the frame of General solvability only? What about 2D or 3D cases?

2. What about the situation $2 < p < 2^*(N, m)$. Is it possible, like in the Laplace equation case, to prove existence of nontrivial generalized solutions?

ACKNOWLEDGMENTS

K.R. Payne has been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The research of N. Popivanov is partially supported by the Bulgarian NSF and Russian NSF under Grant DHTC 01/2/23.06.2017 and by the Sofia University Grant 80-10-216/2017.

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